Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On the Determination of a Certain Class of Surfaces.[*]

BY ARCHER EVERETT YOUNG.

Introduction.

The second fundamental form for a surface S referred to any set of conjugate lines can be written[†]

$$D d\bar{u}^2 + D'' d\bar{v}^2.$$

If the surface in question has positive total curvature there exists an infinity of conjugate systems whose choice as lines of reference, the parameters being properly chosen, reduce this form to

$$D(d\bar{u}^2 + d\bar{v}^2).$$

These particular conjugate systems have been named[‡] isotherm-conjugate. If, on the other hand, the surface has negative total curvature, there exists an infinity of conjugate systems whose choice as lines of reference permit the second form to be written

$$D(d\bar{u}^2 - d\bar{v}^2).$$

We shall, for convenience, refer to such as associate isotherm-conjugate lines. Many well-known surfaces of positive total curvature have *lines of curvature* which belong to the class defined as isotherm-conjugate, while others of negative total curvature have lines of curvature which are associate isotherm-conjugate.

We have discussed in this paper the *problem of determining all surfaces having lines of curvature which are either isotherm-conjugate or associate isotherm-conjugate.*

We show in Article I that on the surfaces having negative total curvature, the asymptotic lines form a network of infinitesimal rhombi, while on those of positive the characteristic lines[§] form a similar network. We may say, therefore, that our *problem is that of determining all surfaces on which either the asymptotic or the characteristic lines form a network of infinitesimal rhombi.*

---

[*] Presented to the American Mathematical Society, December 31, 1915.

[†] Bianchi, "Lezioni di Geometria Differenziale," p. 113.

[‡] Bianchi, *loc. cit.*, p. 168.

[§] A name given by Pucci to that system of lines of positive total curvature, for which when it is parametric, $\frac{D}{E}=\frac{D''}{G}$, $\frac{D}{e}=\frac{D''}{g}$, $D'=0$ ("Dell' angolo caratteristico e delle linee caratterische di una superficie," *Rom. Acc. L. Rend.*, IV (1889), pp. 501-507).

The problem is treated in Article II from the standpoint of the spherical representation of the asymptotic lines for surfaces having negative total curvature, and characteristic lines for surfaces having positive total curvature. After the derivation of the system of equations, on the solution of which the general problem depends, certain particular solutions are considered in Article III. In the closing Article IV, we have reduced the problem to the solution of the fundamental equations used in the Bonnet form, with the addition of another equation in the same functions. We are enabled thus to point out a general method for determining all surfaces of the problem which are solutions of the general "Problem of the Spherical Representation".

I. *The Characteristic Geometric Properties of the Surfaces.*

Let $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ be the expression for the linear element of a surface S of negative total curvature referred to asymptotic lines. The equation of the characteristic lines, which in general is*

$$[D(GD-ED'')-2D'(FD-ED')]du^2+[2D'(GD+ED'')-4FDD'']dudv$$
$$+[2D'(GD'-FD'')-D''(GD-ED'')]dv^2=0,$$

reduces to

$$Edu^2+Gdv^2=0,$$

since the lines of reference are asymptotic. They are, therefore, imaginary on a surface of negative total curvature.

If the asymptotic lines on the surface chosen, form a network of infinitesimal rhombi, the expression above by a proper choice of parameters reduces to

$$ds^2=\lambda(du^2+2\cos\theta\, dudv+dv^2), \qquad \text{(I)}$$

where θ is the angle between the asymptotic lines.

The equations of the characteristic lines and of the lines of curvature on the surface corresponding to (I) are, respectively, $du^2+dv^2=0$, and $du^2-dv^2=0$.

Referring the surface to lines of curvature by writing $du+dv=du_1$, $du-dv_1=dv_1$, (I) becomes† $ds^2=\frac{\lambda}{2}\left((1+\cos\theta)du_1^2+(1-\cos\theta)dv_1^2\right)$, and the equation of the asymptotic lines is now $du_1^2-dv_1^2=0$. Hence the theorem:

---

[*] Eisenhart, *Transactions of A. M. S.*, Vol. V, pp. 421-437.

[†] Knoblauch, "Einleitung in die Allgemeine Theorie der Krummen Flächen," p. 12.

If the asymptotic lines on a surface S of negative total curvature form on the surface a network of infinitesimal rhombi, the lines of curvature must be associate isotherm-conjugate. The converse is easily proven.

From the form of the expression for the linear element just above, it appears that the corresponding surface will be *isothermic*; that is, *have isothermal lines of curvature*, when and only when

$$\tan\frac{\theta}{2}=\frac{f(u_1)}{\phi(v_1)},$$

where, of course, θ is the angle between the asymptotic lines, and hence the theorem:

A surface of negative total curvature which has associate isotherm-conjugate lines of curvature will be isothermic when and only when the tangent of one-half the angle between the asymptotic lines is equal to a function of one parameter divided by a function of the other, the lines of reference being lines of curvature.

Turning now to the consideration of surfaces having positive total curvature, let $d\bar{s}_1^2=E_1d\bar{u}^2+2F_1dvdu+G_1d\bar{v}^2$ be the expression for the linear element of a surface referred to characteristic lines.

Since $\frac{D_1}{E_1}=\frac{D_1''}{G_1}$, $D_1'=0$, the equation of the asymptotic lines may be written thus:

$$E_1\,du^2+G_1\,d\bar{v}^2=0.$$

If the characteristic lines form a network of rhombi, then, by a proper choice of parameters, the expression for the linear element becomes

$$d\bar{s}_1^2=\lambda_1(du^2+2\cos\theta_1\,dudv+d\bar{v}^2), \qquad (\mathrm{I}')$$

where θ_1 is the angle between the characteristic lines. The equations of the asymptotic lines and of the lines of curvature on a surface which corresponds to (I′) are, respectively,

$$d\bar{u}^2+d\bar{v}^2=0, \text{ and } d\bar{u}^2-d\bar{v}^2=0. \qquad (\mathrm{II})$$

Taking the lines of curvature for lines of reference, and following the reasoning of the preceding case, we easily prove the following theorems:

If a surface of positive total curvature has characteristic lines which divide the surface into a network of infinitesimal rhombi, its lines of curvature are isotherm-conjugate; and, conversely.

The necessary and sufficient condition that a surface of positive total curvature, which has isotherm-conjugate lines of curvature, shall be isothermic, is that the tangent of one-half the angle between the characteristic lines shall be equal to a function of one parameter over a function of the other, the lines of reference being lines of curvature.

II. *Discussion of the General Problem.*

If (I) is the first form for a surface S of negative total curvature, referred to asymptotic lines, the third form may be written thus:*

$$d\bar{\sigma}^2=\mu(d\bar{u}^2-2\cos\theta\, du dv+d\bar{v}^2), \tag{III}$$

where $\mu=\dfrac{\lambda}{-K}$, K being the total curvature.

Likewise, if (I′) is the first form for a surface S_1 of positive total curvature, referred to characteristic lines, the third form† is

$$d\bar{\sigma}_1^2=\mu_1(d\bar{u}^2-2\cos\theta_1\, du\, dv+d\bar{v}^2), \tag{III′}$$

where $\mu_1=\lambda_1/K_1$, K_1 as before being the total curvature.

If (III) is the third form for a surface S, referred to asymptotic lines, the functions involved must satisfy the following equation: ‡

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\log\mu}{\partial v}+\cos\theta\dfrac{\partial\log\mu}{\partial u}}{\sin^2\theta}\right]=\frac{\partial}{\partial v}\left[\frac{\partial\log\mu}{\partial u}+\cos\theta\frac{\partial\log\mu}{\partial v}\right]; \tag{1}$$

and, similarly, if (III′) is the third form for a surface S_1 referred to characteristic lines, §

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)+\cos\theta_1\dfrac{\partial(\mu\cos\theta_1)}{\partial v}}{\mu_1\sin^2\theta_1}\right]=\frac{\partial}{\partial v}\left[\frac{\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)+\cos\theta_1\dfrac{\partial(\mu_1\cos\theta_1)}{\partial u}}{\mu_1\sin^2\theta_1}\right]. \tag{2}$$

For the purpose of comparison we write equations (1) and (2) in the following forms, respectively:

$$\begin{aligned}\frac{\partial^2\log\mu}{\partial u^2}-\frac{\partial^2\log\mu}{\partial v^2}+\frac{\partial\log\mu}{\partial u}&\left(\frac{\partial}{\partial u}\log\left(\frac{\cos\theta}{\sin^2\theta}\right)+\frac{\partial}{\partial v}\log\tan^2\frac{\theta}{2}\right)\\ &-\frac{\partial\log\mu}{\partial v}\left(\frac{\partial}{\partial v}\log\left(\frac{\cos\theta}{\sin^2\theta}\right)+\frac{\partial}{\partial u}\log\tan^2\frac{\theta}{2}\right)=0;\end{aligned} \tag{1′}$$

$$\begin{aligned}\frac{\partial^2\log\mu_1}{\partial u^2}-\frac{\partial\log\mu_1}{\partial v^2}+\frac{\partial\log\mu_1}{\partial u}&\left(\frac{\partial}{\partial u}\log\frac{\cos\theta_1}{\sin^2\theta_1}+\frac{\partial}{\partial v}\log\tan^2\frac{\theta_1}{2}\right)\\ &-\frac{\partial\log\mu_1}{\partial v}\left(\frac{\partial}{\partial v}\log\frac{\cos\theta_1}{\sin^2\theta_1}+\frac{\partial}{\partial u}\log\tan^2\frac{\theta_1}{2}\right)\\ &+\frac{\partial^2}{\partial u^2}\log\tan\frac{\theta_1}{2}-\frac{\partial^2}{\partial v^2}\log\tan\frac{\theta_1}{2}=0.\end{aligned} \tag{2′}$$

---

* Bianchi, *loc. cit.*, p. 156. † Bianchi, *loc. cit.*, p. 168. ‡ *Ibid.*, p 156. § *Ibid.*, p. 169.

The Gauss equation which the functions appearing in (1′) and (2′), respectively, must satisfy, may be written thus:

$$\csc^2\frac{\theta}{2}\left(\frac{\partial^2\log\mu}{\partial u^2}+\frac{\partial^2\log\mu}{\partial v^2}\right)-2\left(1-\cot^2\frac{\theta}{2}\right)\frac{\partial^2\log\mu}{\partial u\partial v}-\frac{\partial}{\partial u}\log\tan^2\frac{\theta}{2}\frac{\partial\log\mu}{\partial u}$$
$$-\frac{\partial}{\partial v}\log\tan^2\frac{\theta}{2}\frac{\partial\log\mu}{\partial v}+8\cos^2\frac{\theta}{2}\mu-4\cot\frac{\theta}{2}\frac{\partial^2\theta}{\partial u\partial v}=0, \tag{3}$$

where μ and θ are to be replaced, respectively, by μ_1 and θ_1 when used with (2′).

The general problem, then, is reduced to the solutions of the pairs of equations (1′), (3) *and* (2′), (3); *the first leading to the surfaces of negative total curvature which are solutions of the problem, and the second to those of positive.*

As the general solution of these systems of equations can not be obtained, the general solution of the problem, from this standpoint, at least, is denied us. Particular solutions are, however, easily obtained, and the method of procedure is as follows:

Having obtained a solution of the pair of equations (1′), (3) or (2′), (3), we have the third form for the corresponding surfaces referred, in the first case to asymptotic lines, and in the second to characteristic lines. The first forms, or the expressions for the linear elements on the surfaces, are then obtained by quadrature from one or the other of the following systems of equations:[*]

$$\left\{\begin{aligned}\frac{\partial\log\rho}{\partial u}&=\frac{-\cos\theta\dfrac{\partial\log\mu}{\partial v}-\dfrac{\partial\log\mu}{\partial u}}{\sin^2\theta}\\ \frac{\partial\log\rho}{\partial v}&=\frac{-\cos\theta\dfrac{\partial\log\mu}{\partial u}-\dfrac{\partial\log\mu}{\partial v}}{\sin^2\theta}\end{aligned}\right\}, \tag{4}$$

where $K=-1/\rho^2$;

$$\left\{\begin{aligned}\frac{\partial\log\rho_1}{\partial u}&=\frac{\cos\theta_1\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)+\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)}{\mu_1\sin^2\theta_1}\\ \frac{\partial\log\rho_1}{\partial v}&=\frac{\cos\theta_1\dfrac{\partial}{\partial v}(\mu_1\cos\theta_4)+\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)}{\mu_1\sin^2\theta_1}\end{aligned}\right\} \tag{4′}$$

where $K_1=1/\rho^2$.

---

[*] Bianchi, *loc. cit.*, p. 155 and p. 168.

The Cartesian coordinates may then be obtained, following the general method, by the integration of a Ricatti equation or by some special device, depending upon the functions involved.

Before closing this article we shall call attention to some particular solutions which are suggested by the form of the equations (1), (2) and (3).

CASE 1. If in particular the function μ reduces to a constant, the corresponding surfaces are pseudo-spherical. All pseudo-spherical surfaces are included here. Likewise, if the function μ_1 reduces to a constant, the integration of the equations (2′) and (3) leads to the surfaces of constant positive total curvature.

CASE 2. If $\cos\theta = f(u+v)$, equation (1′) may be written in the form

$$\frac{\partial^2 \log\mu}{\partial u^2} - \frac{\partial^2 \log\mu}{\partial v^2} - \phi(u+v)\left(\frac{\partial \log\mu}{\partial u} - \frac{\partial \log\mu}{\partial v}\right) = 0, \tag{5}$$

where ϕ being a function of θ must be a function of $u+v$ as indicated. The general solution of (5) is $\log\mu = F(u+v)\psi(u-v) + \chi(u+v)$, where ψ and χ are arbitrary functions, and $\phi = 2F'/F$.

Substituting this value for $\log\mu$ in the Gauss equation (3), changing parameters by writing $u+v=u_1$, $u-v=v_1$, differentiating, and separating functions of u_1 from functions of v_1, we show that the function ψ must reduce to a constant. It follows easily that the corresponding surfaces are surfaces of revolution. All surfaces of revolution having negative total curvature are included here. In a similar way, we obtain the surfaces of revolution having positive total curvature, from the solution of (2′) and (3), assuming that the $\cos\theta_1$ is a function of the sum of the two variables.

If, in particular, θ is a constant, the integration of (1′), (3) and (4) gives as the linear element of the corresponding surfaces the following:

$$d\bar{s}^2 = \left[\frac{(1-be^{c(u+v)})^2}{e^{c(u+v)}}\right]^{\tan^2\frac{\theta}{2}} (d\bar{u}^2 + 2\cos\theta\, du dv + d\bar{v}^2), \tag{I}$$

where b and c are constants and θ is the angle between the asymptotic lines. The surfaces corresponding to (I) are the surfaces of revolution of negative total curvature which have the same spherical representation of their lines of curvature as the catenoid or minimal surface of revolution. To every value of θ from 0 to π, there corresponds a separate surface. Beside the catenoid corresponding to $\theta = \frac{\pi}{2}$ we may mention a surface generated by revolving the parabola about a line perpendicular to its axis, and also a surface generated by a fourth degree curve having a conjugate point, as being included in the set.

Likewise, if θ_1 is a constant, we have as the first form for the corresponding surfaces

$$d\bar{s}_1^2 = \left[\frac{(1-be^{c(u+v)})^2}{e^{c(u+v)}}\right]^{-\tan^2\frac{\theta_1}{2}} (du^2 + 2\cos\theta_1\, du dv + d\bar{v}^2), \qquad (\mathrm{I}')$$

where θ_1 is now the angle between the characteristic lines of course.

These compose the class of surfaces of revolution, of positive total curvature, which have the same spherical representation of their lines of curvature as the catenoid.

CASE 3. An examination of equations (1′) and (2′) shows that if $\theta_1 = \theta$, and the function θ is chosen so that

$$\frac{\partial^2 \log \tan \theta/2}{\partial u^2} = \frac{\partial^2 \log \tan \theta/2}{\partial v^2},$$

the equations which μ and μ_1 must satisfy are one and the same; and, hence the theorem:

To every surface S of negative total curvature which is a solution of the problem and which has the tangent of one-half the angle between the asymptotic lines equal to a function of one of the parameters divided by a function of the other, the lines of reference being lines of curvature, there exists another S_1 of positive total curvature, having the same spherical representation of its characteristic lines as S of its asymptotic.

It is easily shown that any two surfaces S and S_1 which are related in this way are isothermic surfaces, connected by the Bour-Darboux theorem.[*] It is easily proven: *That all isothermic surfaces S and S_1 connected by the Bour-Darboux theorem are associates of one another, the asymptotic lines of one corresponding to the characteristic lines on the other, and vice versa.*[†] The converse of this theorem is true.

---

[*] Darboux, "Théorie des Surfaces," Vol. II, p. 243.

[†] We have discussed in detail in a previous paper the problem of determining such isothermic surfaces.

IV. *Discussion of the Problem from the Standpoint of the Spherical Representation of the Lines of Curvature.*

Taking the Bonnet form* for the fundamental equation of a surface referred to lines of curvature, we have

$$\left.\begin{aligned} \frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ&=0,\\ \frac{\partial P}{\partial v}+MQ&=0,\\ \frac{\partial Q}{\partial u}-NP&=0,\end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} M&=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},\\ N&=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u},\end{aligned}\right\} \tag{7}$$

the first and third form for the surface being respectively,

$$d\bar{s}^2=Ed\bar{u}^2+Gd\bar{v}^2, \tag{I}$$

$$d\bar{\sigma}^2=P^2\,du^2+Q^2\,dv^2. \tag{III}$$

For the surfaces sought, the parameters being properly chosen, $D=\pm D''$, and hence,

$$\frac{E}{G}=\frac{Q^2}{P^2} \text{ or } \frac{\sqrt{E}}{\sqrt{G}}=\pm\frac{Q}{P}, \tag{8}$$

the plus and minus signs corresponding respectively to surfaces of positive and negative curvature.

Eliminating G from (7) by means of (8), we have

$$\left.\begin{aligned} \frac{\partial\log\sqrt{E}}{\partial v}&=\mp\frac{P}{Q}M,\\ \frac{\partial\log\sqrt{E}}{\partial u}&=-\frac{\partial\log\left(\frac{P}{Q}\right)}{\partial u}\pm N\frac{Q}{P}.\end{aligned}\right\} \tag{9}$$

The condition for the integrability of (9) may be written thus

$$\frac{\partial^2\log\left(\frac{P}{Q}\right)}{\partial u\partial v}\mp\frac{\partial}{\partial u}\left(\frac{MP}{Q}\right)\mp\frac{\partial}{\partial v}\left(N\frac{Q}{P}\right)=0, \tag{10}$$

---

* *Journal de l'École Polytechnique,* Vol. XLII, pp. 132–151.

where the upper signs in both (9) and (10) are for surfaces of positive total curvature, the lower for those of negative.

This equation determines the lines on the Gauss sphere which permit of surfaces of positive or negative total curvature of the class sought, having them as the spherical representation of their lines of curvature.

Evidently the two cases reduce to one and the same when $\frac{\partial^2 \log}{\partial u \partial v}\left(\frac{P}{Q}\right)=0$; that is, when $\frac{P}{Q}=\frac{f(u)}{\phi(v)}$.

The corresponding surfaces are the isothermic surfaces discussed in the preceding article.

If we make the substitution

$$\frac{P}{Q}=\frac{1}{\phi}\left(\mp\int N\phi\,du+V\right),$$

(10) transforms into the Laplace equation

$$\frac{\partial^2\phi}{\partial u\partial v}-\frac{\partial \log M}{\partial u}\frac{\partial\phi}{\partial v}+MN\phi=0. \tag{11}$$

Eliminating Q from the last two equations of (6) we have again equation (11) with ϕ replaced by P.

The problem, then, is reduced to the simultaneous solution of the following system of equations:

$$\left.\begin{aligned}
&\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ=0,\\
&\frac{\partial Q}{\partial u}-NP=0,\\
&\frac{\partial^2 P}{\partial u\partial v}-\frac{\partial \log M}{\partial u}\frac{\partial P}{\partial v}+MNP=0,\\
&\frac{P}{Q}=\frac{1}{P_1}\left(\mp\int NP_1\,du+V\right),
\end{aligned}\right\} \tag{12}$$

where P_1 is any solution of the preceding Laplace equation. Given any set of values M, N, P, Q, satisfying this system of equations, we have, by quadrature, from (8) and (9), the functions appearing in the first fundamental form of surfaces which are solutions of the problem. The Cartesian coordinates for any point on the same can be obtained by quadrature as soon as the Cartesian coordinates of the corresponding point on the Gauss sphere are known.

The problem of solving the first three equations of (12) is identical with that of the "Problem of the Spherical Representation" as treated by us in a previous article.* If we add any fourth equation to these three we can expect, of course, a corresponding solution as the number of equations then only equals the number of functions to be determined. Considering the third equation as a Laplace equation for the determination of P we are led to choose as the new equation some one of the following expressions equated to zero.

$$\begin{aligned} h &= -MN, \\ k &= -\frac{d^2 \log M}{dudv} - MN, \\ k_1 &= -\frac{d^2 \log (Mk)}{dudv} - k, \\ k_2 &\equiv -\frac{\partial^2 \log (M k k_1)}{\partial u \partial v} - k_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_i &= -\frac{\partial^2 (\log Mk \ldots . k_{i-1})}{\partial u \partial v} - k_{i-1}, \end{aligned}$$

where i is an arbitrarily large positive integer.

The general solution of the Laplace equation can be found when and only when h or k_i vanishes.† If h vanishes, neglecting the case where the corresponding surfaces are developable, we may say that M vanishes. The corresponding surfaces are those on which the lines $v=$const. are geodesics. The corresponding lines on the Gauss sphere are great circles. The problem of determining these surfaces, then, is solved as soon as one determines the unit sphere referred to systems of great circles and their orthogonal trajectories.

If, however, h does not vanish but k does, the corresponding surfaces are either surfaces for which one set of the lines of curvature, the lines $v=$const., have constant geodesic curvature, or certain surfaces having the same spherical representation of their lines of curvature as these.‡ Moreover, all these surfaces can be obtained by algebraic methods and quadrature from the general class corresponding to the case where h vanishes.§

And, in general, the surfaces corresponding to the case $k_{i+1}=0$ can always be obtained by algebra and quadrature from those corresponding to the preceding case $k_i=0$.

The totality of surfaces found in this way, starting with the case $M=0$ and continuing to the case $k_i=0$, where i is arbitrarily large make up the general class belonging to the problem of the spherical representation.

---

* A. E. Young, "On the Problem of the Spherical Representation," etc., AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, No. 1.

† See previous reference, p. 42. ‡ A. E. Young, *loc. cit.*, p. 47. § *Loc. cit.*, p. 43.

The surfaces of this class belonging to our present problem can be found as follows:

Taking the general class corresponding to the case $M=0$, we choose the corresponding functions E, G, P and Q so as to satisfy the relation *

$$\frac{E}{G}=\frac{UQ^2}{VP^2}, \tag{13}$$

where U and V are functions of u and v, respectively, to be determined. The original class with these limitations on the functions are solutions of the problem, and, in fact, compose that system of surfaces belonging to the problem on which one set of the lines of curvature are geodesics.

Again, taking the general class of surfaces corresponding to $k=0$, we may choose the functions P and Q corresponding, so that the last equation of (12) is satisfied, P_1 now being a general solution of the Laplace equation and hence actually known, and thus obtain at once the surface which we seek from the general class. Or, following the preceding method, we may choose the new functions E, G, P and Q corresponding to the case $k=0$ so that they satisfy (13) and thus obtain the desired surfaces. The surfaces in this case will be those of the problem having one set of lines of curvature characterized by geodesic curvature, and certain other surfaces having the same spherical representation of their lines of curvature as these.

In general, therefore, the surfaces of the problem, involved in the solution of the "Problem of the Spherical Representation," can be obtained from the separate systems as classified above by a process which involves nothing more difficult than the limitation by the simple algebraic equation (13) of the four functions appearing in the first and third fundamental quadratic forms, the lines of reference being lines of curvature.

MIAMI UNIVERSITY.

---

* The introduction of the functions U and V is due to the fact that no particular choice of parameters has been made, as was done when equation (8) was derived.